CONTROL SYSTEM DESIGN

Module – I (12 Hours)

Introduction: Application of software and simulink for control system design, Review of compensation technique and choice of optimum parameters to obtain desired performance, Absolute stability and relative stability concepts. Design of Linear Control Systems: Transient and steady state response; Polar, Bode, Root locus plots; Reshaping of these plots to obtain desired response, Initial condition and forced response, A simple lag – lead design.

Module – II (14 Hours)

Design of Control Systems by State Variable Techniques: Controllability, Observability; Stability by using computer methods; solution of state and output equations of closed loop systems. Pole placement design, Observer design. Linear / quadratic optimal control. Full and reduced order observers. Design of Nonlinear Control Systems: Phase plane technique, Describing Function method for nonlinearities like saturation, dead space, ON/OFF (Ideal Relay type nonlinearity). Simulation techniques.

Module – III (14 Hours)

PID Controller: Use of digital computer as a compensator device, basic computer control scheme, tunable PID controller, Ziegler – Nichol's method, Simulation of multiloop control system using P, PI, PD, PID controller and finding the system response. Standard compensator structures (P, PD, PI and PID control). Design of Digital Control System: Technique and methodology; Computation of digital equivalent of the analog controller, simulation of performance of the design. Digital controller design, Regulator and observer design; Digital servo for inverted pendulum.

Textbooks:

1. G. C. Goodwin, S. F. Graebe, M. E. Salgado, Control System Design, Prentice Hall of India, 2001.
2. George Ellis, Control System Design Guide – A Practical Guide, 3rd Edition, Academic Press, 2005 Indian Reprint, ISBN: 81-8147-596-8.
3. Norman S. Nise, Control Systems Engineering, 3rd Edition, Wiley.

Recommended Reading:

1. M. Gopal, Digital Control and State Variable Method, Tata McGraw Hill.
2. Hadi Saadat, Computational Aids in Control System Using MATLAB, McGraw Hill International.
3. Ogata K., Modern Control Engineering, 4th Edition, Prentice Hall
4. Ogata K. System Dynamics, 3rd Edition, Prentice Hall
5. M. Gopal, Control Systems Principles and Design, 2nd Edition, Tata McGraw Hill

Stability — All roots (poles of the syst) of denominator of transfer function must lies in the left half of s-plane.

Types
1. Absolute Stability ② conditional Stability
3- BIBO Stability ④ Asymptotic Stability
⑤ Relative Stability

Absolute Stability
- Parameter doesnot changed.
- The term absolute stability is used in relation to qualitative analysis of stability.
- The absolute stability can be determined from the location of the roots of the characteristic equation in s-plane.

Conditional Stability
- It is stable in certain condition.
- can change in certain parameters. but not in all parameters.

BIBO Stability
I/P has certain magnitude just like impulse. When we apply finite i/p then we get finite o/p.

Asymptotic Stability

It relates to the transient condition. If the syst. o/p dies out with initial condition having no. physical i/p.

Relative Stability

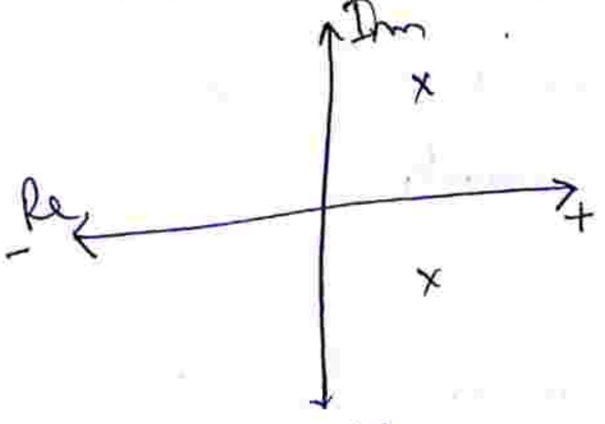
- Relative stability used in relation to comparative analysis of stability.
- Max. overshoot, damping ratio, Gain Margin, phase margin are measurements of relative stability.

Unstable

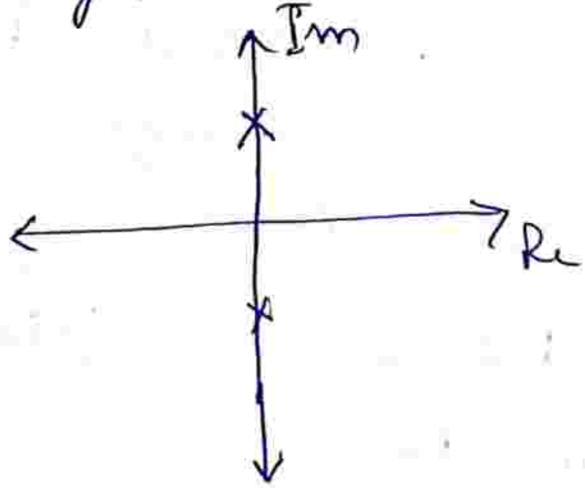
If any of roots or whole roots lies in R.H.S of s-plane, then it is unstable.

Marginally Stable

→ If any poles or roots of the system lies on imaginary axis.



(Unstable)

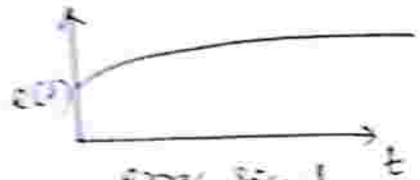


(Marginal Stable)

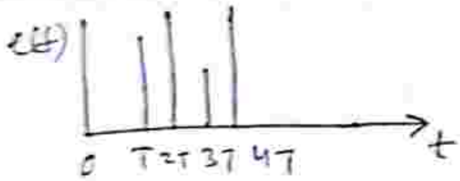
Sample Date Control Systems

* The control Systems using one or more Signals at discrete time intervals are known as sampled data control Systems.

* The control Signal is amplitude Sampled train pulses and the control Systems employing such Signals are known as Sampled State control System.

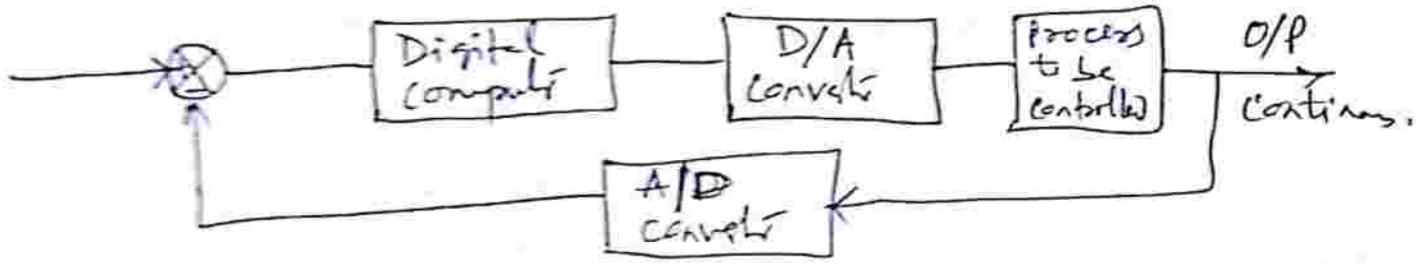


error Signal

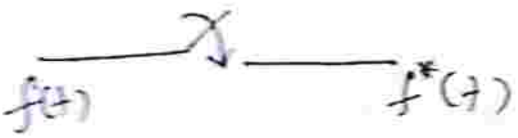


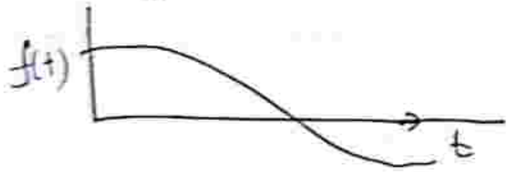
Sampled Error Signal

* A digital CS makes use of a digital computer and the actuating signal is in the form of a coded ~~signals~~ digital data at discrete intervals of time.



The on-off temperature control is a simplest example of Sampled - data control System.





→ The basic element of a Sample - data Control System is a Sampler which Samples the continuous Signal into a Sequence pulses appearing at regular interval of time.

→ The Sampled O/P denoted by $f^*(t)$ is known as starred function of $f(t)$.

$$\left[f^*(t)\right]^* = f^*(t)$$

Sampling process

→ The sampler converts a continuous signal into a sequence pulses wherein the magnitude of the pulse gives the value of input signal at the instant of sampling.

Properties of Z-Transform

→ Delayed function $Z[f(t-KT)] = Z^{-k} Z[f(t)] = Z^{-k} F(z)$

→ Multiplication by t

$$Z[t f(t)] = -Tz \frac{d}{dz} Z[f(t)] = -Tz \frac{d}{dz} F(z)$$

T → Sampling time

→ Division by t

$$Z\left[\frac{1}{t} f(t)\right] = -\frac{1}{T} \int \frac{Z[f(t)]}{z} dz$$

$$= -\frac{1}{T} \int \frac{F(z)}{z} dz$$

→ $\boxed{Z[e^{\pm at} f(t)] = F(z e^{\pm aT})}$

→ Final value theorem

$$f(\infty) = \lim_{Z \to 1} (1 - z^{-1}) F(Z)$$

* $(1-z^{-1}) F(z)$ has no pole on or outside the unit circle in z-plane having origin as its centre.

→ Initial value theorem

$$f(0) = \lim_{z \to \infty} F(z)$$

This is the strength of the first sample $f(0)$.

$$\boxed{Z[f(t)] = F(Z) = \sum_{k=0}^{\infty} f(kT) z^{-k}}$$

$k = 0, 1, 2, 3 \ldots$

$$F(z) = f(0) z^{0} + f(T) z^{-1} + f(2T) z^{-2} + f(3T) z^{-3} + \ldots$$
$$\ldots + f(kT) z^{-k}.$$

Difference equations

→ The analysis of Sampled-data control system can be carried out in terms of difference equations because the i/p to the system is in the form of a signal occurring at regular intervals of time.

* In terms of z-transform, the output function $C(Z)$ is given by

$$C(Z) = \frac{G(Z)}{1+G(Z)H(Z)} R(Z)$$

$$\Rightarrow \frac{C(Z)}{R(Z)} = \frac{G(Z)}{1+G(Z)H(Z)} = \text{Pulse transfer function}$$

* The z-transform is related to difference equation in a similar manner as the Laplace transform to differential equation.

→ The Laplace transform converts ~~linear~~ linear differential with constant co-efficients into algebraic equation in terms of "s".

→ Similarly, Z-transform converts difference equation with constant co-efficients into algebraic equation in terms of "z".

In general

$$\boxed{Z\left[x(k+n)\right] = z^n X(z) - z^n x(0) - z^{n-1} x(1) - \cdots - z x(n-1)}$$

Where n being a +ve integer.

* When a difference equation is transformed into an algebraic one, the initial data are already included in the algebraic representation.

Hold circuit

→ The sampled signal provided by the sampling process is a weighted impulse train $f^*(t)$ which being the input signal to the transfer function.

* The original signal is reconstructed from the sampled signal by using a hold circuit.
  i) The hold circuit brings smoothness in the sampled output.
  ii) Hold circuit enables to hold the signal between two consecutive sampling instant at the preceded value till the next sampling instant is reached.

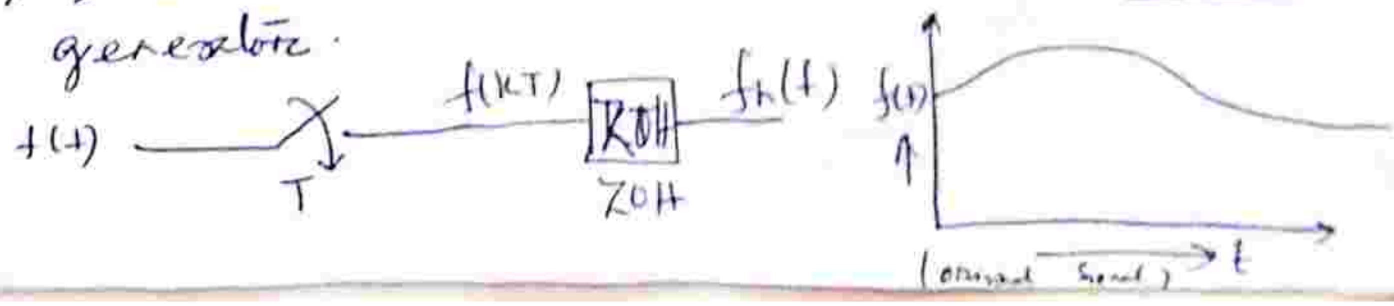
* At the sampling instants the hold signal and original signal have the same value.

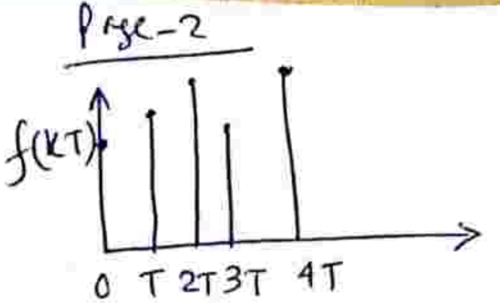
Zero-order Hold

* Hold signal is the zeroth derivative of an impulse.

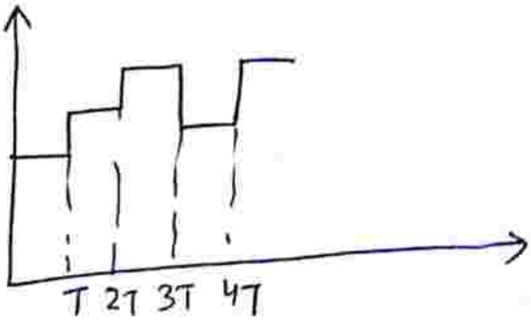
⇒ The hold circuit holds the output signal at a fixed level between two consecutive sampling instants such that the slope of the hold circuit signal is zero or in other words hold signal is the zeroth-derivative of an impulse. Such a holding device is called zero-order hold (ZOH).

→ The ZOH device is also known as "box-car" generator.

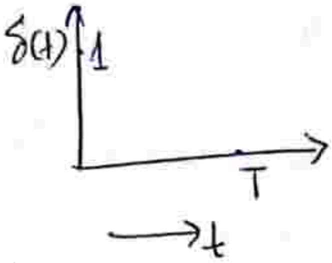


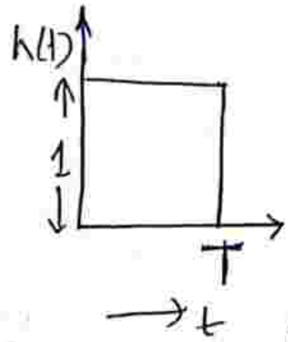


(Sampled signal)



(ZOH O/P)





$$i/p \longrightarrow \boxed{ZOH} \longrightarrow O/P$$

(Unit impulse to ZOH)

→ The O/P ZOH is a unit step function appearing upto time 'T', so we can write

$$h(t) = u(t) - u(t-T)$$

$$L\{u(t)\} = \frac{1}{s}$$

$$L\{u(t-T)\} = \frac{1}{s} e^{-sT}$$

$$L\{h(t)\} = L\{u(t) - u(t-T)\}$$

$$= 1/s - 1/s \, e^{-sT}$$

$$H(s) = 1/s - 1/s \, e^{-sT}$$

→ As input to ZOH is $\delta(t)$ i.e. unit impulse function, the Laplace transform of the input is $L\{\delta(t)\} = 1$

→ The TF of the ZOH is denoted by $G_{ho}(s)$

$$G_{ho}(s) = \frac{L(\text{o/p of ZOH})}{L(\text{i/p of ZOH})} = \frac{1/s - 1/s \, e^{-sT}}{1}$$

$$G_{ho}(s) = \frac{1}{s}(1 - e^{-sT}) = \boxed{\frac{1-e^{-sT}}{s}}$$

Gain frequency plot of ZOH circuit

→ The Tf of ZOH is given by

$$G_{ho}(s) = \frac{1-e^{-sT}}{s}$$

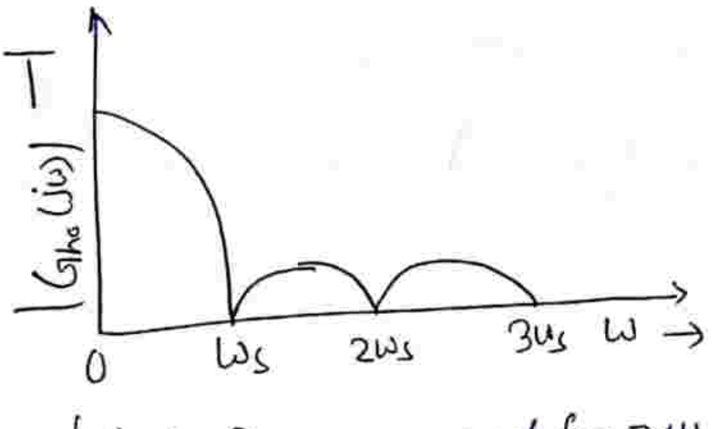
→ The Sinusoidal form of above Tf is obtained by putting $s = j\omega$

$$\Rightarrow G_{ho}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}$$

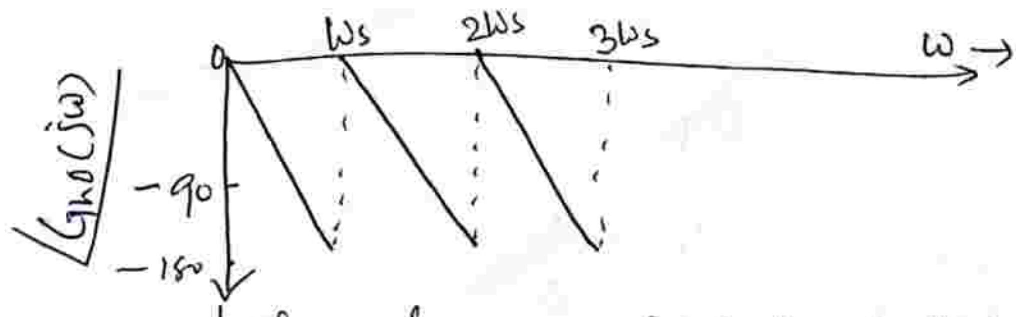
$$|G_{ho}(j\omega)| = T\left|\frac{\sin\left(\frac{\pi\omega}{\omega_s}\right)}{\left(\pi\omega/\omega_s\right)}\right|$$

$$\angle G_{ho}(j\omega) = -\left(\frac{\pi\omega}{\omega_s}\right) + \theta \quad \text{where } \theta = 0,\ \sin\left(\frac{\pi\omega}{\omega_s}\right) > 0$$

$$\theta = \pi,\ \sin\left(\pi\omega/\omega_s\right) < 0$$



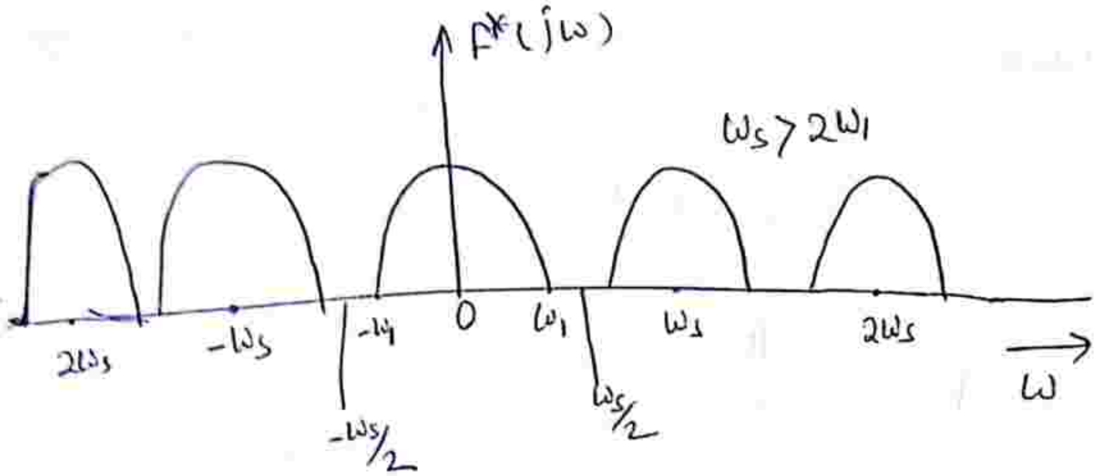
(Gain frequency graph for ZOH)



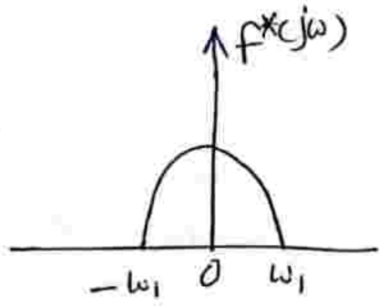
(phase-frequency graph for ZOH)

Reconstruction of Signal: Minimum Sampling frequency

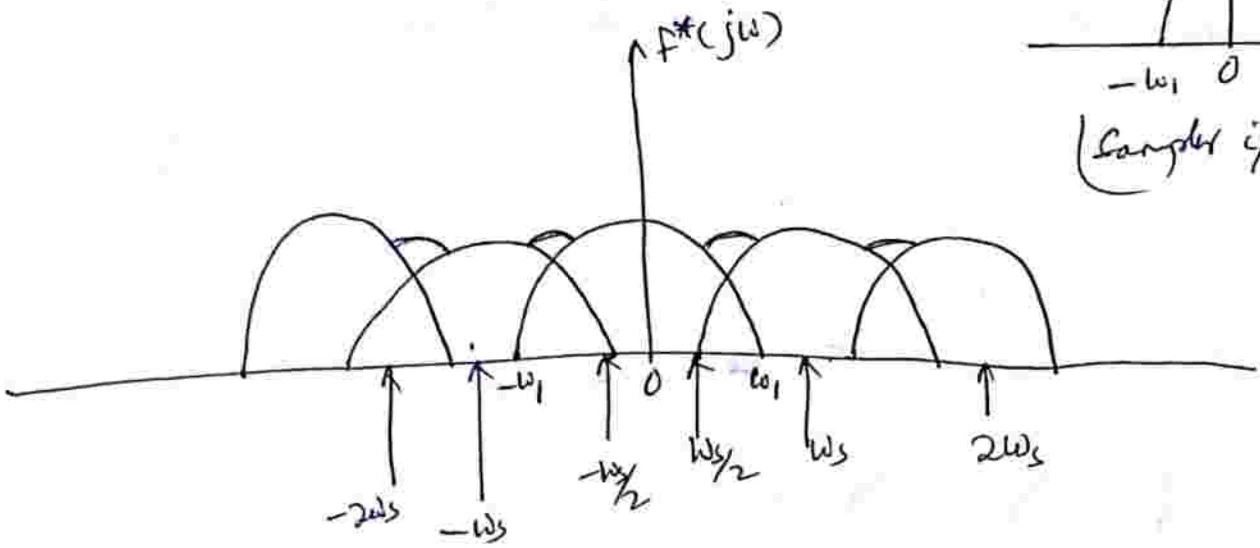
→ The use of Sampler in control System generates high frequency components in the Sampled output.



Sampler o/p frequency $\omega_s > 2\omega_1$



(Sampler i/p frequency)



Sampler o/p frequency $\omega_s < 2\omega_1$

* $\omega_s \geq 2\omega_1$ → Sampling theorem

ω_s ↓ Sampling frequency ω_1 → input frequency

Compensator

→ Main objective of compensator is to neutralise deficiency of a system.

Systems are two types

Stable → The system needs to improve performance - (In case of type 0/1 system)

Unstable - The first objective is to bring ~~bri~~ the system to stable.

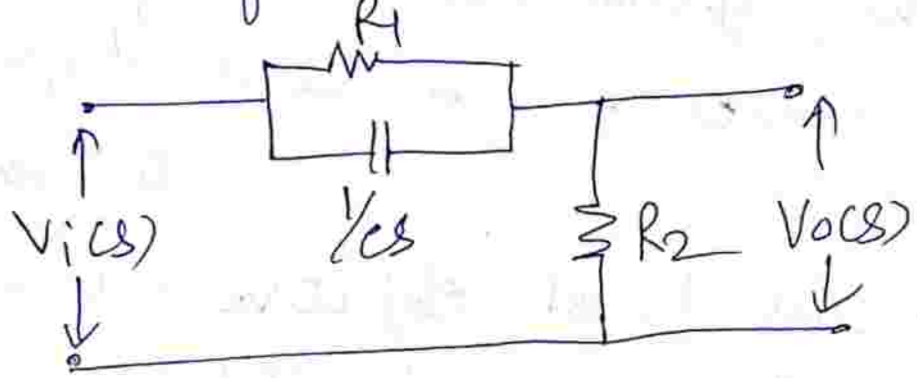
Type-2 → Unstable → Specifically need lead compensator.

Type-0/1 → Stable - It may need Lag/Lead/Lag-Lead compensator.

Lead compensator

→ Lead compensator mostly provides phase-lead to the system.

* The transfer function can be realised in the form of electrical N/W



→ Using voltage-divider rule

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2}{R_2 + \left(R_1 \parallel \frac{1}{Cs}\right)}$$

$$= \frac{R_2}{R_2 + \dfrac{R_1 \cdot 1/Cs}{R_1 + \frac{1}{Cs}}}$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{R_2\left(R_1 + \frac{1}{Cs}\right)}{R_1R_2 + \left(R_2 + R_1\right)\frac{1}{Cs}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2}{R_1+R_2}\left(\frac{1+R_1Cs}{1+\frac{R_1R_2}{R_1+R_2}Cs}\right)$$

Attenuation constant $= \alpha = \frac{R_2}{R_1+R_2}$

Where ($\alpha < 1$) and time constant $\boxed{\tau = R_1C}$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{\alpha(1+\tau s)}{1+\alpha\tau s} \quad \text{--- (1)}$$

Zeros at $s = -1/T$ and pole at $s = \frac{1}{-\alpha T}$

The TF of equn(1) is in sinusoidal form

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{\alpha(1+j\omega\tau)}{(1+j\omega\alpha\tau)}$$

phase lead to syst. O/P.

Maximum phase-Shift

The phase lead provided by lead compensat

$$\phi_m = \tan^{-1}\omega_m\tau_1 - \tan^{-1}\omega_m\alpha\tau_1 \quad — ① \text{ (rep}$$

$$\phi_m = \tan^{-1}\omega_m\tau_1 - \tan^{-1}\omega_m\alpha\tau_m \quad — ①$$

Using condition $\frac{d\phi}{d\omega} = 0$, we find that the max. phase lead occurs at

$$\omega_m = \sqrt{\frac{1}{\tau} \times \frac{1}{\alpha\tau}} = \frac{1}{\sqrt{\alpha}} \cdot \frac{1}{\tau} = \frac{1}{\tau}\frac{1}{\sqrt{\alpha}}$$

($\omega_m = \frac{1}{\tau\sqrt{\alpha}}$ = max phase shift at this freq)

To get max. phase lead, put $\omega_m = \frac{1}{\tau\sqrt{\alpha}}$ in phase lead expression

$$\phi_m = \tan^{-1}\omega_m\tau - \tan^{-1}\omega_m\alpha\tau \quad — ②$$

put the value of ω_m in eqn (2), get

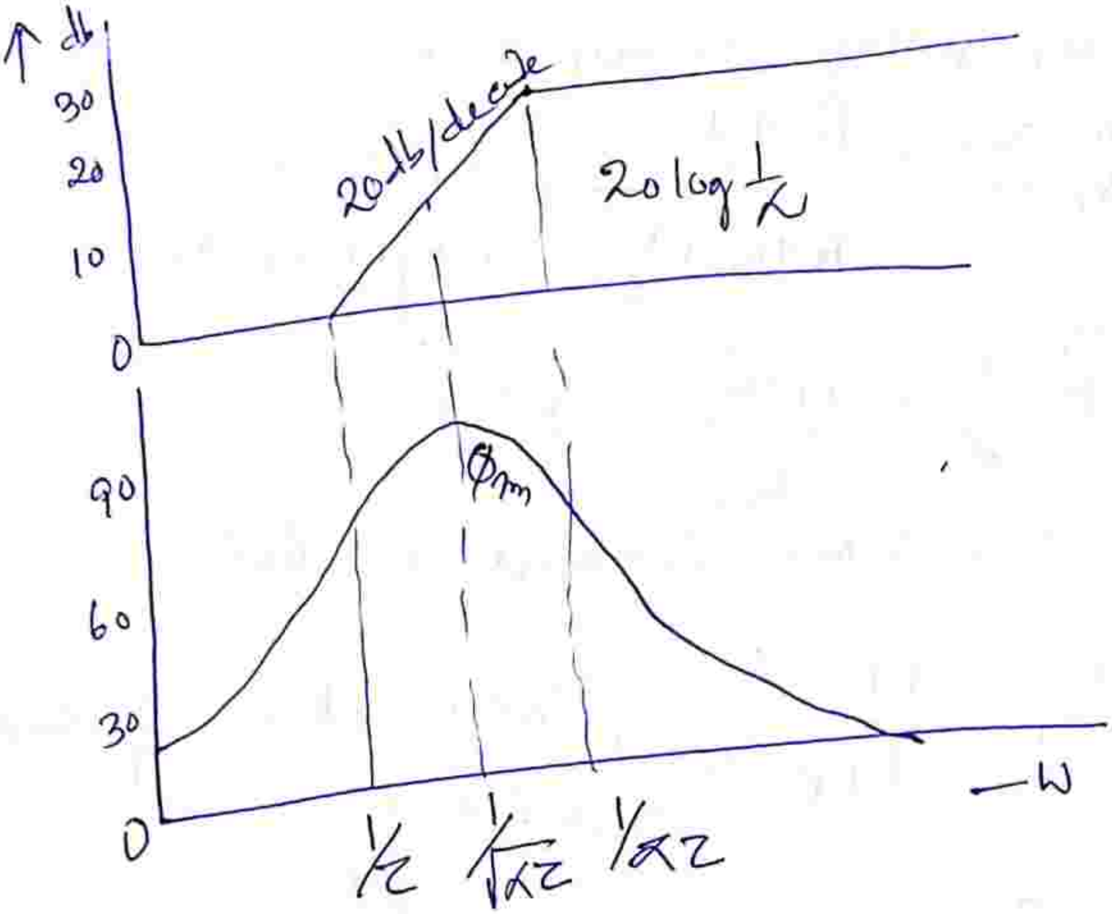
$$\phi_m = \tan^{-1}\frac{1}{\sqrt{\alpha}} - \tan^{-1}\sqrt{\alpha}$$

$$\phi_m = \tan^{-1}\frac{1-\alpha}{2\sqrt{\alpha}}$$

$\Rightarrow \tan\phi_m = \dfrac{1-\alpha}{2\sqrt{\alpha}}$, so $\sin\phi_m = \dfrac{1-\alpha}{1+\alpha}$

$$\boxed{\phi_m = \sin^{-1}\frac{1-\alpha}{1+\alpha}} \qquad \boxed{\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}}$$

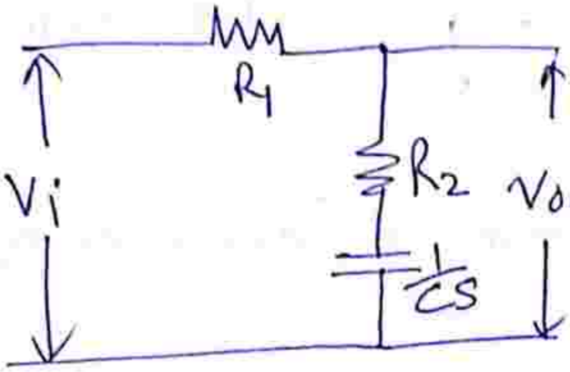
Bode plot design of lead compensator



$\therefore$ $\alpha < 1$, this ϕ_m is of leading in nature as shown in bode plot.

Lag Compensator

→ It improves the steady-state response of the system i.e reduce the ess of the system.



Using voltage divider rule

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} = \frac{1 + R_2 C s}{1 + \left(\frac{R_1 + R_2}{R_2}\right) C s}$$

Put $\beta = \frac{1}{\alpha} = \frac{R_1 + R_2}{R_2}$ $(\beta > 1)$

and time constant $\tau = R_2 C$

$$\frac{V_o(s)}{V_i(s)} = \frac{1 + \tau s}{1 + \beta \tau s},$$ zeros at $s = -\frac{1}{T}$ and poles at $s = -\frac{1}{\beta T}$

→ The phase lag provided by lag compensator

$$\phi_m = \tan^{-1} \omega_m \tau - \tan^{-1} \omega_m \beta \tau$$

Using condition $\frac{d\phi}{d\beta} = 0$, we find that the max. phase lag occurs at freq.

$$\omega_m = \sqrt{\frac{1}{\beta \tau} \times \frac{1}{\tau}} = \frac{1}{\tau\sqrt{\beta}}$$

$$\boxed{\omega_m = \frac{1}{\tau\sqrt{\beta}}}$$

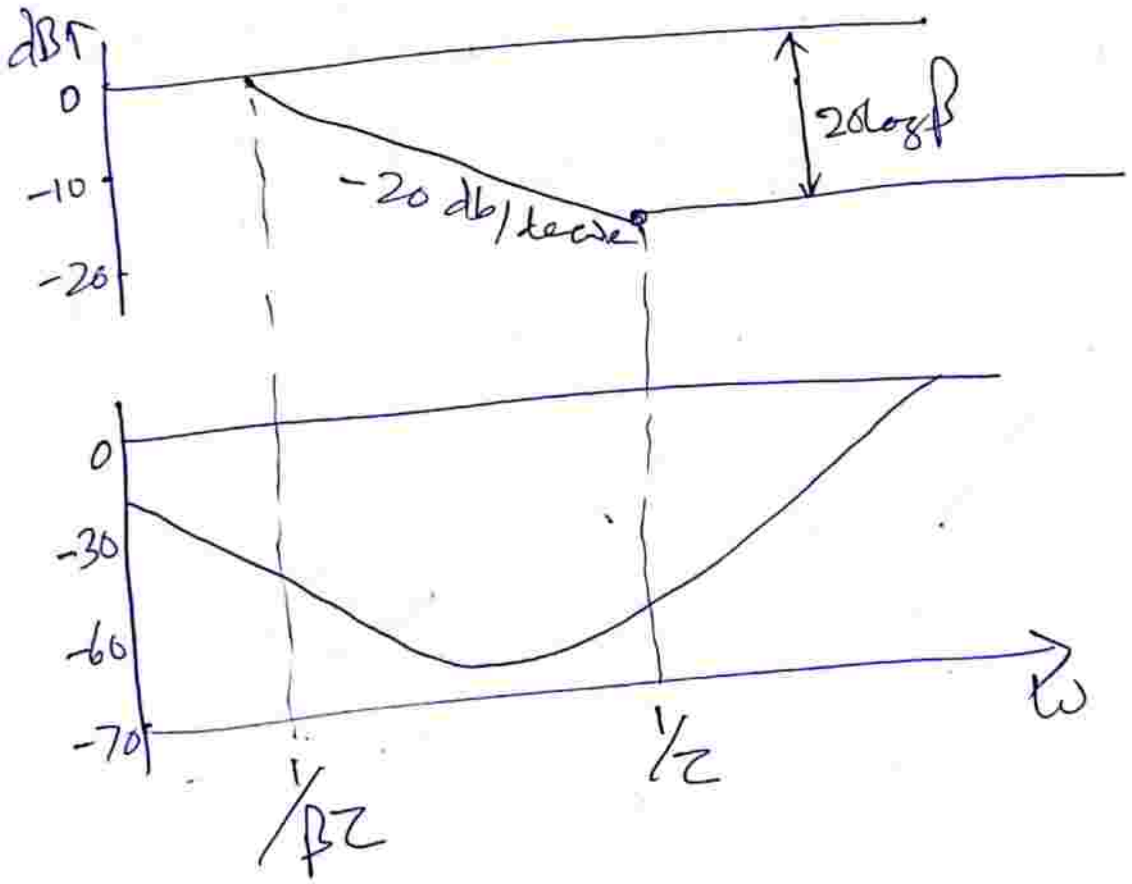
$$\phi = \tan^{-1}\left(\frac{1-\beta}{2\sqrt{\beta}}\right) \quad \text{as } \phi_m = \tan^{-1}\omega_m \tau - \tan^{-1}\omega_m \beta\tau$$

$$\sin\phi = \frac{1-\beta}{1+\beta} \quad \text{or} \quad \beta = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

$\therefore$ $\beta > 1$, this ϕ_m is of lagging in nature

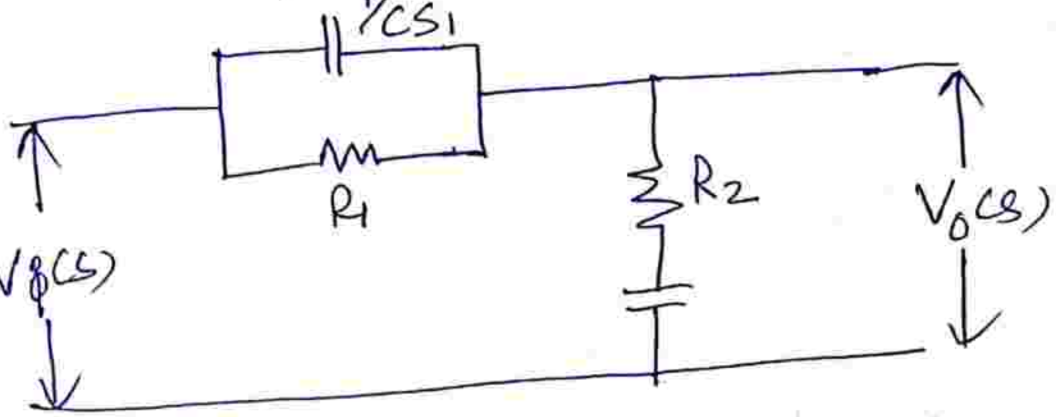
So, pole is dominant over zero.

Bode plot



Lag-Lead compensator

→ It is used for improving both transient and Steady-State response of the System.



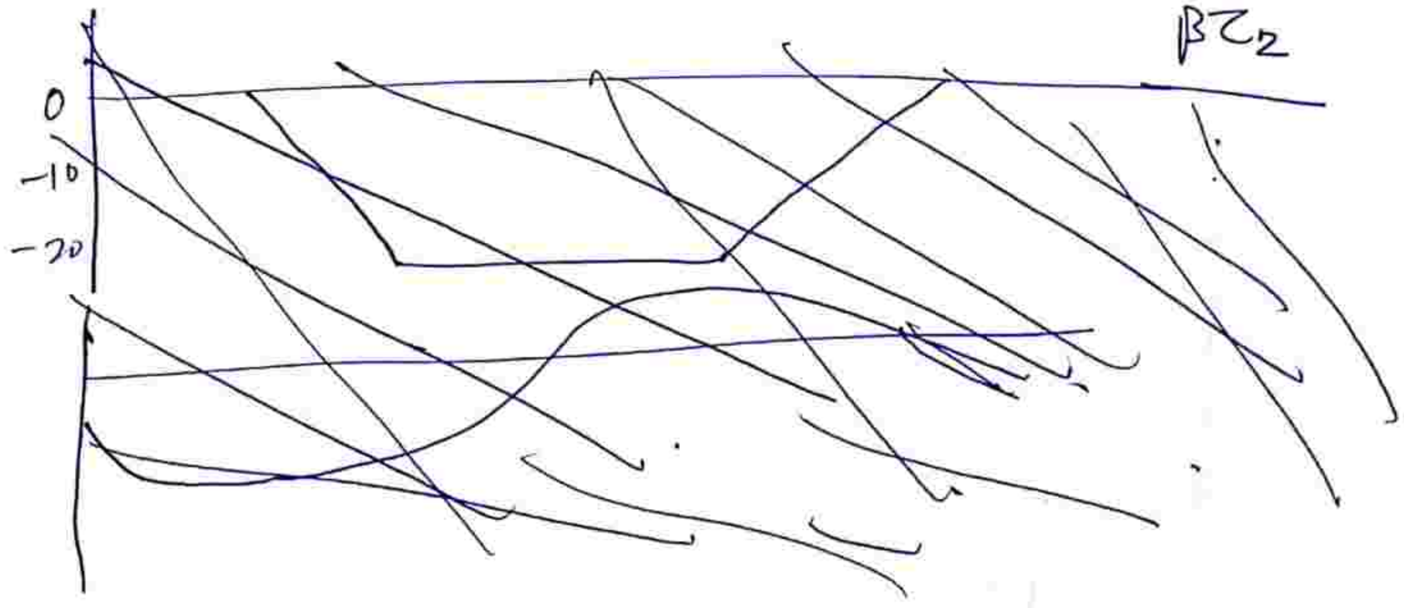
$$\frac{V_0(s)}{V_i(s)} = \frac{\alpha(1+\tau_1 s)(1+\tau_2 s)}{(1+\tau_1 s)(1+\beta\tau_2 s)}$$

where $\alpha = \frac{R_2}{R_1+R_2}$ $\beta = \frac{1}{\alpha} = \frac{R_1+R_2}{R_2}$

$\tau_1 = R_1C_1$ → time constant for lead N/W

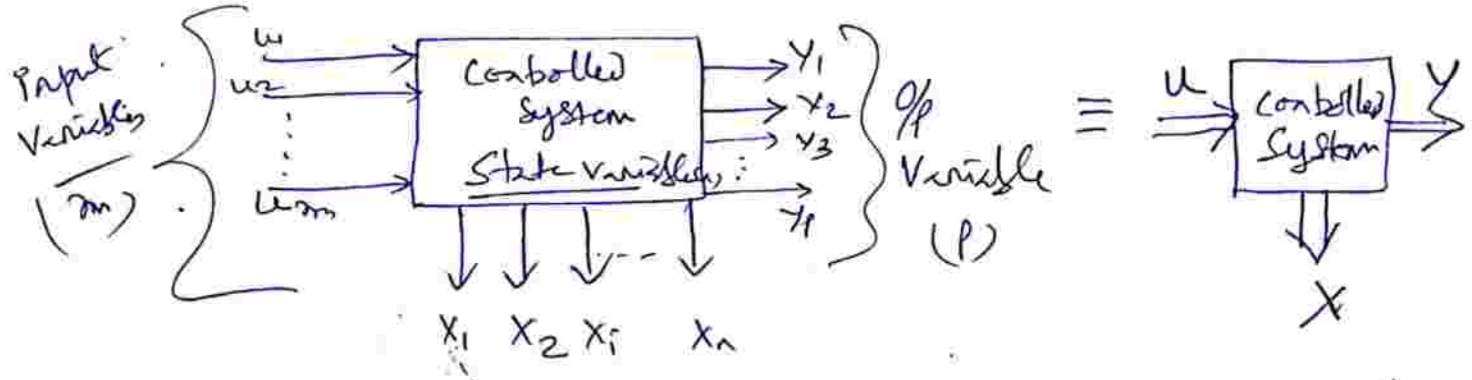
$\tau_2 = R_2C_2$ → time constant for lag N/W.

Zeros at $s = -\frac{1}{\tau_1}$ & $-\frac{1}{\tau_2}$ Poles at $s = -\frac{1}{\alpha\tau_1}$ & $-\frac{1}{\beta\tau_2}$



State variable analysis and Design

* A mathematical abstraction to represent or model the dynamics of a system utilises three types of variables called the input, the output and the state variables.
$u(t)$ $y(t)$ $x(t)$



Structure of a general control System

* The state of a dynamical system is a minimal set of variables (known as state variables) such that the knowledge of these variables at $t=t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behavior of the system for $t > t_0$.

* The state vector x determines a point (called the state point) in an n-dimensional space, called the state space.

* The curve traced out by the state point from $t=t_0$ to $t=t_1$ in the direction of increasing time is known as the state trajectory.

$Y(t) = g(x(t), u(t))$; time invariant systems

$Y(t) = g(x(t), u(t), t)$; time varying systems.

* The state and output equations constitute the state model of the system.

① State Space representation of Systems

→ The nth order differential equation is given by

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = u(t) \quad \text{— (1)}$$

i/p = u(t) and initial conditions

$y(0), \frac{dy(0)}{dt} \ldots \frac{d^{n-1}y(0)}{dt^{n-1}}$ at $t=0$ are given

→ The dynamic behavior of differential equ^n (1) can be determined from the knowledge of $u(t)$, $y(t) \ldots y^{n-1}(t)$.

$y(t), \dot{y}(t), \ldots y^{n-1}(t)$ can be considered in terms of a set of 'n' state variables.

Let $y = x_1$, $\dot{y} = x_2$

. . . .

$y^{n-1} = x_n$

→ The set of equations (1) can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\boxed{\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \cdots - a_1 x_{n-1} + u}$$

General representation of State equation

$$\dot{X} = AX + Bu$$

Where X = State vector $(n \times 1)$ $\quad u$ = control (Scalar)
A = Matrix $(n \times n)$ $\quad B$ = Matrix $(n \times 1)$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; \quad A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 \cdots & 1 & 0 \\ \cdots & & & & \\ -a_n & -a_{n-1} & \cdot & \theta & \cdot a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The output equation is given by

$$y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

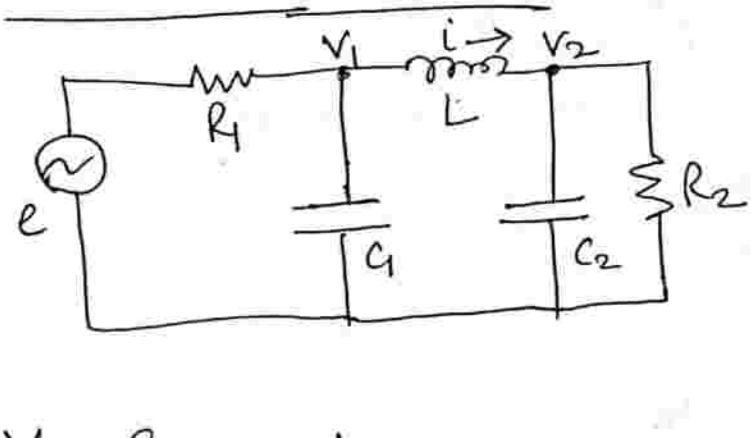
$$y = Cx$$

Where $C = [1 \; 0 \; 0 \cdots 0]$, y = O/P (Scalar)

C = matrix $(1 \times n)$

The O/P equation is algebraic and State equation is a Set of first order differential equation

- electrical network



initial values

$i(0), V_1(0), V_2(0)$

are known.

State variables

$V_1(t), V_2(t), i(t)$

$$\frac{V_1 - e}{R} + C\frac{dV_1}{dt} + i = 0 \quad \text{--- (1)}$$

$$C_2\frac{dV_2}{dt} + \frac{V_2}{R_2} - i = 0 \quad \text{--- (2)}$$

$$L\frac{di}{dt} + V_2 - V_1 = 0$$

$\frac{dv_1}{dt} = ?$ $\frac{dv_2}{dt} = ?$ $di/dt = ?$

v_1, v_2, i are selected as state variables.

$x_1 = v_1$, $x_2 = v_2$, $x_3 = i$

$$\dot{x}_1 = -\frac{1}{R_1C_1}x_1 - \frac{1}{C_1}x_3 + \frac{1}{R_1C_1}e$$

$$\dot{x}_2 = -\frac{1}{R_2C_2}x_2 + \frac{1}{C_2}x_3$$

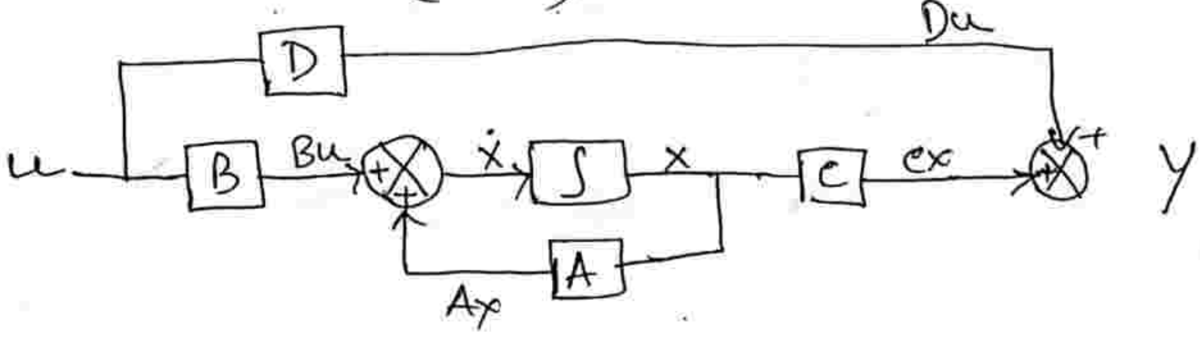
$$\dot{x}_3 = \frac{1}{L}x_1 - \frac{1}{L}x_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1/R_1C_1 & 0 & -1/C_1\\ 0 & -1/R_2C_2 & 1/C_2\\ 1/L & -1/L & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1/R_1C_1\\ 0\\ 0\end{bmatrix}e$$

Block diagram for State equation

$\dot{X} = AX + Bu$ $y = Cx + Du$

$\dot{x}$ = State vector ($n\times1$)
u = Control vector ($m\times1$)
y = O/P vector ($P\times1$)
A = matrix ($n\times n$)
B = matrix ($n\times m$)
C = matrix ($P\times n$)
D = matrix ($P\times m$)



Transfer Function Decomposition

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2+2s+5}$$

transform to time domain, the differential equation is obtained

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = u$$

$$\boxed{\begin{aligned} y &= x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -5x_1 - 2x_2 + u \end{aligned}}$$

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Solution of State equation

→ $\frac{dx}{dt} = ax$; $x(0) = x_0$ classical method of

Soln by considering 1st order differential equation.

* This equation has the soln

$$x(t) = e^{at} x_0 = \left(1 + at + \frac{1}{2!}a^2t^2 + \cdots + \frac{1}{i!}a^it^i + \cdots\right) x_0$$

$$\dot{x}(t) = Ax(t) ; \quad x(0) = x_0 \quad \text{——— (1)}$$

It represents homogeneous linear system (unforced system) with constant co-efficients.

→ By analogy with scalar case, we assume a soln of the form $x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_i t^i + \cdots$

where a_i are the vector co-efficients.

* By substituting the assumed soln in eqn (1) we get

$$a_1 + 2a_2 t + 3a_3 t^2 + \cdots = A(a_0 + a_1 t + a_2 t^2 + \cdots)$$

The comparision of vector co-efficient of equal powers of t, yields $a_1 = Aa_0$, $a_2 = \frac{1}{2}Aa_1 = \frac{1}{2!}A^2 a_0$

$$\vdots \qquad \vdots$$

$$a_i = \frac{1}{i!}A^i a_0$$

In the assumed soln, equating $x(t=0) = x_0$, we find that $a_0 = x_0$.

The soln of $x(t)$ is found to be

$$x(t) = \left(I + At + \frac{1}{2!}A^2t^2 + \cdots \frac{1}{i!}A^it^i + \cdots\right) x_0$$

(2) $e^{At} = \left(I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{i!}A^it^i + \cdots\right)$

The soln $x(t)$ can be written as

$$\boxed{x(t) = e^{At}x_0}$$

$e^{At} \rightarrow$ known as State transition matrix, It is denoted by $\phi(t)$.

* Soln of non-homogeneous State eqn (Forced System)

$$\dot{x}(t) = Ax(t) + Bu(t) \quad x(0) = x_0$$

$$\Rightarrow \dot{x}(t) - Ax(t) = Bu(t)$$

Multiplying both sides by e^{-At}, we can write

$$\underbrace{e^{-At}\left[\dot{x}(t) - Ax(t)\right]} = e^{-At}Bu(t)$$

$$\| $$

$$\frac{d}{dt}\left[e^{-At}x(t)\right]$$

Integrate both sides w.r.t 't' limit 0 and t

$$e^{-At}x(t)\Big]_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

Premultiplying both sides by e^{At}, we have

$$x(t) = \underbrace{e^{At}x(0)}_{\text{Homogeneous}} + \underbrace{\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{Forced soln}}$$

If the initial state soln is known at $t = t_0$ rather than t_0

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)\,d\tau.$$

Transfer Matrix

The matrix relating Laplace transform of O/P Laplace transform of i/p of State Space representation of a control system is known as transfer matrix.

→ The state equations of a system are given below

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= cx + Du \end{aligned}\right\} \text{——— (1)}$$

where x - State vector $(n\times 1)$ $u \rightarrow$ i/p (Scalar)

and $y \rightarrow$ O/P (Scalar) $A \rightarrow$ Matrix $(n\times n)$

$C \rightarrow$ matrix $(1\times n)$ $D \rightarrow$ Scalar

Taking Laplace transform on both sides (1)

$$\left.\begin{aligned} SX(S) - X(0) &= AX(S) + BU(S) \\ Y(S) &= CX(S) + DU(S) \end{aligned}\right\} \text{——— (2)}$$

According to the definition T.f $\underline{X(0) = 0}$

$$\left.\begin{aligned} SX(S) &= AX(S) + BU(S) \\ \cancel{Y(S)} &\cancel{= CX(S) + DU(S)} \\ X(S) &= (SI - A)^{-1} BU(S) \end{aligned}\right\} \text{——— (3)}$$

Substituting (3) in (2), the following equation is obtained

$$Y(S) = C(SI - A)^{-1} BU(S) + DU(S)$$

$$Y(S) = [C(SI - A)^{-1}B + D]\, U(S)$$

The transfer matrix is $G(S) = \frac{Y(S)}{U(S)}$

$$= C(SI - A)^{-1}B + D$$

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Part-2

The transfer function of MIMO System

$$\boxed{G(s) = C(SI-A)^{-1}B + D}$$

Controllability

The controllability is in relation to transfer of a system from one state to another by appropriate input controls in a finite time.

* A system is said to be controllable, if it is possible to have an input u to transfer the system from any given initial state $x(t_0)$ to any given final state $x(t_f)$ over specified interval of time $(t_f - t_0)$.

$$\dot{x} = Ax + Bu$$
$$y = cx + Du$$

$x(0) \rightarrow$ initial state
$x(n) \rightarrow$ final state

$$\dot{x}(k+1) = A(\cancel{k}x) + Bu(k)$$

* A system is said to be observable if every state x_0 can be exactly determined from the measurement of the O/P y over a finite interval of time $0 < t < t_f$.

Four type of States

→ System is controllable and observable (SCO)
→ " is controllable and un-observable (SCU)
→ System is un-controllable and observable (SUO)
→ The System is un-controllable and un-observable (SUU)

Kalman Test

→ A linear time invariant continous system described by the state equation

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is completly controllable if and only if the rank of the controllability matrix is defined as

$$Q_c = [B : AB : A^2B : \dots . A^{n-1}B]$$ is equal to rank 'n'.

* The system represented by the equation is observable if and only if $(n \times n_m)$ composite matrix is obtained

$$Q_o = [C^T : A^TC^T : (A^T)^2C^T : \dots (A^T)^{n-1}C^T]$$

$\boxed{\text{How to find out the rank of the matrix?}}$

Duality property

1) The pair (AB) is controllable implies that the pair A^TB^T is observable
2) The pair (AC) is observable implies that the pair (A^TC^T) is controllable

* The concepts of controllability and observability are dual concepts.

Properties of State transition Matrix

1) $\phi(0) = e^{A.0} = I$ 2. $\phi(t) = e^{At}$

$$\phi^{-1}(t) = \phi(-t)$$

3. $\phi(t_1+t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = \phi(t_1) \cdot \phi(t_2)$

$$= \phi(t_2) \cdot \phi(t_1)$$

$$\boxed{\phi(s) = (sI-A)^{-1} \Rightarrow \text{resolvent matrix}}$$

Cayley Hamilton Theorem

* Every Square matrix satisfies its own characteristics equation.

$$q(\lambda) = |\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n = 0$$

→ the formal procedure of evaluation of the matrix polynomial $f(A)$ is given below

① find the eigenvalues of matrix A

② If all the eigenvalues are distinct, solve 'n' simultaneous eqn. for the co-efficients $\alpha_0, \alpha_1 \dots \alpha_{n-1}$

If A possess an eigenvalues λ_k of order 'm', then only one Independent equation can be obtained by substituting λ_k in $f(\lambda_i) = R(\lambda_i)$ $i = 1, 2, 3 \dots$. The remaining $(m-1)$ linear equation find $f(A) = e^{At}$ for $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$q(\lambda) = |\lambda I - A| = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} = (\lambda+1)^2 = 0$$

$$\boxed{\lambda_1, \lambda_2 = -1}$$

Since A is a second order, the polynomial $R(\lambda)$ will be of the following form

$$R(\lambda) = \alpha_0 + \alpha_1 \lambda$$

→ The coefficient α_0 & α_1 are evaluated for

$$f(-1) = f(\lambda_1) = e^{-t} = \alpha_0 - \alpha_1 = R(-1)$$

$$\left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=-1} = t e^{-t}$$

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Eigen Vectors of nxn non-singular matrix (1)

→ The vector x_i is termed to be an eigen vector of matrix A associated with eigen values λ_i $(i=1,2,3\ldots)$ and satisfies following equation

$$\lambda_i x_i = A x_i$$

$$\lambda_i x_i - A x_i = 0$$

$$|\lambda I - A| x_i = 0 \quad \text{——— (1)}$$

The eigen vectors of matrix A are determined by solving equation (1).

* The eigen vectors of the matrix $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

The characteristics equation of matrix A is given by $|\lambda I - A| = 0$

So, $\lambda_1 = -1$ and $\lambda_2 = 5$

(i) The eigen vector x_1 associated with eigen value $\lambda_1 = -1$ is obtained by solving following equation.

$$|\lambda_1 I - A| x_1 = 0$$

Put $\lambda_1 = -1$ and $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

$$\left\{(-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -4 & -4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = 0$$

$$-4m_{11} - 4m_{21} = 0$$

and $-2m_{11} - 2m_{21} = 0$

Select $m_{11} = 1$

then $m_{21} = -1$

Then the eigen vector associated with eigen value $\lambda_1 = -1$ is $x_1 = \begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

ii) The eigen vector associated with eigen value $\lambda_2 = 5$ is obtained by solving following equation

$$|\lambda_2 I - A| X_2 = 0$$

Put $\lambda_2 = 5$ and $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$ in the above equation

$$\left\{5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}\right\}\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = 0$$

$$2m_{12} - 4m_{22} = 0$$

$$-2m_{12} + 4m_{22} = 0$$

Select/choose $m_{22} = 1$

then $m_{12} = 2$

The eigen vector is associated with eigen value $\lambda_2 = 5$ is $x_2 = \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The matrix formed by the eigen vectors as its columns is known as modal matrix of matrix 'A' and denoted as M.

$$\text{Model matrix}(M) = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ \vdots & & & & \\ m_{n1} & m_{n2} & \cdots & \cdots & m_{nn} \end{bmatrix}$$

The model matrix 'M' is used to diagonalise the matrix 'A'.

Diagonalisation of nxn matrix

The matrix having elements only on its diagonal is known as diagonal matrix. The inverse of a diagonal matrix is obtained by

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}^{-1} = \begin{bmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/\beta & 0 \\ 0 & 0 & 1/\gamma \end{bmatrix}$$

* The state model having elements of co-efficient matrix A only on its diagonal is called canonical state model which is obtained by parallel decomposition of transfer function.

* General state model can be transformed into canonical form using the process diagonalisation.

→ nth order MIMO System

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \quad (1)$$

Where A is $n \times n$ non-singular matrix with distinct eigenvalues.

Assume a new state variable z such that

$$X = Mz \text{ ——— } (2)$$

Where, M being $(n \times n)$ non-singular constant matrix

on substituting equ^n (2) into state equ^n (1)

Then $M\dot{z} = AMz + Bu$ ——— (3)

$$y = CMz + Du \text{ ——— } (4)$$

$$\dot{z} = M^{-1}AMz + M^{-1}Bu \text{ ——— } (5)$$

Select matrix M such that matrix $M^{-1}AM$ is transformed into a diagonal matrix. The matrix $M^{-1}AM$ is denoted as Λ.

So $\boxed{\Lambda = M^{-1}AM}$ → diagonal matrix

The matrix is a diagonal matrix, the diagonal elements of which are given by distinct eigenvalues of ~~nxn~~ non-singular $(n \times n)$ matrix.

Bush form or phase variable form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & & a_n \end{bmatrix}$$

* The phase variable form of matrix A has a unit element above its diagonal and -ve co-efficient element in the last row. All other elements being zero.

The modal matrix with reference to Bush's or phase variable form of matrix 'A' is denoted 'P' and given by

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \\ \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

Where $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are distinct eigen values of matrix 'A' which is given in Bush or phase variable form.

* The matrix $\Lambda = P^{-1}AP$ is a diagonal one. The matrix 'P' is known as Vandermonde's matrix.

* For diagonalisation $(n \times n)$ matrix having distinct eigen values and the matrix given in Bush's form or phase variable form, then ~~B~~ use of Vandermonde matrix is suitable.

Controllability & observability

Controllability → A system is said to be controllable if the state $x(t_f)$ is determined or measured from the initial conditions $x(t_0)$ and i/p $u(t)$.

$$\dot{x} = Ax(t) + Bu(t), \quad y = Cx + Du$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \quad A \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

$$x(t_f) = f\{x(t_0), u(t)\}$$

Where $x(t_f)$ → state vector and finite time where $(t_f > 0)$

x → State vector, u = i/p vector

y = o/p vector and A, B, C, D are matrix.

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}_{n\times n} \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}_{n \times n} \qquad D = \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & D_{2n} \\ \vdots & & & \\ D_{n1} & D_{n2} & \cdots & D_{nn} \end{bmatrix}_{n \times n}$$

condition of controllability

$$Q_c = [B, AB, A^2B \cdots A^{n-1}B]$$

Where $A_{n \times n}$ matrix

* If Rank of controllability of matrix (Q_c) is 'n' then system is controllable.

If Rank $R=3$, then all states are controllable

If $R=2$, then 2-states are controllable and one is not.

If $R=1$, then 2-states are controllable and one is not.

If $R=0$, then not controllable.

Observability

A system is said to be observable if its initial state $x(t_0)$ can be observed from its O/P $y(t_0)$ over a finite interval of time (t_f).

$$x(t_0) = f\{y(t_f)\}$$

$$Q_0 = [C^T, A^T C^T, (A^T)^2 C^T \dots (A^T)^{n-1} C^T]$$

n = order of the system

Q_0 = observability test matrix

For $\dot{x} = Ax + Bu$, $y = Cx$.

Principle of Duality

- If AB is controllable, then $A^T B^T$ is observable
- If AC is observable, then $A^T C^T$ is controllable

Ex – Test the observability

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1] x.$$

Pole assignment / pole placement Design

→ Devlop a design procedure for generating control laws generally known as pole assignment or pole placement when the system dynamics is represented in state-variable form.

* The design procedure is concerned with assignment of the poles of the closed loop transfer function to any desired locations in the stable zone of the plane of the roots by using state variable form.

⇒ In general, an n^{th} order system is modeled by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{—— (1)}$$

and the control i/p $u(t)$ is generated as a linear combination of the states as

$$u(t) = -K'x(t) \quad \text{—— (2)}$$

$$\text{where } K' = [k_1 \; k_2 \cdots k_n] \quad \text{—— (3)}$$

The equation (1) can be written as

$$\dot{x}(t) = (A - BK')\,x(t) \quad \text{—— (4)}$$

* We choose the desired pole locations

$$S_i = \lambda_i, \quad i = 1 \text{ to } n \quad \text{—— (5)}$$

Then the closed loop characteristics equation is

$$\alpha_c(s) = |sI - A + BK'| = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n) \quad \text{—— (6)}$$

→ In this equation there are 'n' unknowns, $K_1, K_2 \cdots$ and 'n' known co-efficients in the right hand side of eqn (6).

→ We can solve for the unknown gains on the left hand side by equating co-efficients of equal powers of 's' on the right hand side in equation (6).

Ackermann's formula

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{——— (1)}$$

→ A matrix polynomial is formed using the co-efficients of the desired characteristics polynomial

$$\alpha_c(s) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \quad \text{——— (2)}$$

→ According Cayley-Hamilton theorem, a matrix satisfies its own characteristics equation.

So, replacing 's' with the matrix 'A' in equation (2), we get

$$\alpha_c(A) = A^n + \alpha_{n-1} A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I \quad \text{—— (3}$$

Then Ackermann's formula for the gain matrix K is given by

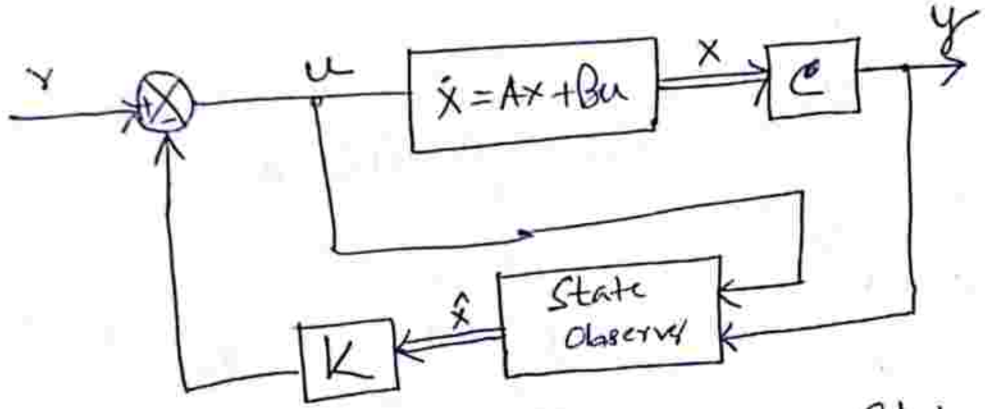
$$K' = [0\ 0\ 0 \cdots 1]\left[B\ AB \cdots A^{n-2}B\ A^{n-1}B\right]^{-1} \alpha_c(A)$$

* Note: $\left[B\ AB \cdots A^{n-2}B\ A^{n-1}B\right]$ is the controllability matrix and Ackermann's formula will produce a result if the system is controllable.

Observer Systems

→ For pole-placement by state-feedback, the control system law is $\boxed{u = -Kx + r}$

* All state variables are accessible for measurement and control purposes. Only inputs and outputs can be used to drive a device whose o/p will approximate the state vector. This device is called State-observer.



A linear system with state observer

Ex

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} u = Ax + Bu$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x = Cx$$

Design observer.

Soln To verify the system is observable.

$$|\lambda I - A| = 0 \quad \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \lambda-1 & -2 & 0 \\ -3 & \lambda+1 & -1 \\ 0 & -2 & -\lambda \end{bmatrix} = 0$$

$$(\lambda-1)\left[-\lambda(\lambda+1) - 2\right] + 2(3\lambda) = 0$$

$$\Rightarrow (\lambda-1)(-\lambda^2-\lambda-2) + 6\lambda = 0$$

Check $\boxed{\lambda_1 = -4 \quad \lambda_2, \lambda_3 = -3 \pm j1}$

Let $G = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$ Then $|A - GC|$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}_{3\times1} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}_{1\times3}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & g_1 \\ 0 & 0 & g_2 \\ 0 & 0 & g_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -g_1 \\ 3 & 1 & 1-g_2 \\ 0 & 2 & -g_3 \end{bmatrix}$$

The characteristics eqution $|\lambda I - (A - GC)| = 0$

$$\begin{bmatrix} \lambda-1 & -2 & g_1 \\ -3 & \lambda+1 & g_2-1 \\ 0 & -2 & \lambda+g_3 \end{bmatrix} = 0$$

Then, it gives $\lambda^3 + g_3\lambda^2 + (2g_2 - 9)\lambda + 2 + 6g_1 - 2g_2 - 7g_3 = 0$ ... -- eqn — (1)

The desired characteristics equation is

$$(\lambda+3+j1)(\lambda+3-j1)(\lambda+4)=0$$

$$\Rightarrow \lambda^3+10\lambda^2+34\lambda+40=0 \quad \cdots (2)$$

By comparing the equation (1) & (2)

$$\lambda^3+g_3\lambda^2+(2g_2-9)\lambda+2+6g_1-2g_2-7g_3=0$$

$\cancel{\lambda^3+10\lambda^2}$

$$((\lambda+3)^2+1)(\lambda+4)=0$$

$$(\lambda^2+9+6\lambda+1)(\lambda+4)=(\lambda^2+6\lambda+10)(\lambda+4)$$

$$=\lambda^3+4\lambda^2+6\lambda^2+24\lambda+10\lambda+40$$

$$=\lambda^3+10\lambda^2+34\lambda+40$$

$$\boxed{\begin{gathered} g_3\lambda^2=10\lambda^2 \\ g_3=10 \end{gathered}}$$

$$\boxed{\begin{gathered} 2g_2-9=34 \\ g_2=\frac{34+9}{2}=\frac{43}{2}=21.5 \end{gathered}}$$

$$\boxed{\begin{gathered} 2+6g_1-2g_2-7g_3=40 \\ \Rightarrow 6g_1=40-2+2g_2+7g_3=38+2\times 21.5+7\times 10 \\ 6g_1=38+43+70=151 \\ g_1=\frac{151}{6}=25.2 \end{gathered}}$$

$$G=\begin{bmatrix} 25.2 \\ 21.5 \\ 10 \end{bmatrix}$$

Reduced order Observer

→ To develop the design equations for the reduced order observer, we partition the state vector

$$x(t) = \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix}$$

where $x_a(t)$ are the states to be measured and $x_b(t)$ are the states to be estimated.

→ $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + \cancel{Du(t)}$, this plant equation can be partitioned as

$$\begin{bmatrix} \dot{x}_a(t) \\ \dot{x}_b(t) \end{bmatrix} = \begin{bmatrix} A_{aa} & A_{ab} \\ A_{ba} & A_{bb} \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix} + \begin{bmatrix} B_a \\ B_b \end{bmatrix} u(t) \quad \text{--- (1)}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_a(t) \\ x_b(t) \end{bmatrix}$$

→ The equations for the measured states can be written as

$$\dot{x}_a(t) = A_{aa}x_a(t) + A_{ab}x_b(t) + B_a u(t) \quad \text{--- (2)}$$

$$\Rightarrow \dot{x}_a(t) - A_{aa}x_a(t) - B_a u(t) = A_{ab}x_b(t) \quad \text{--- (3)}$$

$$\dot{x}_b(t) = A_{ba}x_a(t) + A_{bb}x_b(t) + B_b u(t)$$

$$\Rightarrow \dot{x}_b(t) = A_{bb}x_b(t) + \left[A_{ba}x_a(t) + B_b u(t)\right] \quad \text{--- (4)}$$

→ For the reduced order observer, we consider the LHS of equation (2) to be the "known measurements" and the terms in equn (4) $A_{ba}x_a(t) + B_b u(t)$ is considered to be "known inputs".

~~No~~

Non-linear System

Every real control System is non-linear and in every linear analysis and design method, we have to describe the use of linear approximation to the real models.

Types of Non-linear System

- Describing Function Analysis
- phase plane Analysis
- Lyapunov Stability Analysis.

* Describing function Analysis method is approximate extension of frequency response method (including Nyquist Stability criterion) to Non-linear Systems.

* Phase-plane method has basic idea to solve second order differential equation and graphically display the result as a family of system motion trajectories on a two dimensional plane, called the phase-plane which allows use to visually observe the motion patterns of the systems.

* Lyapunov Stability Analysis — It is applicable to all kinds of control System with hard or soft non-linearities and second order or higher order. It is applicable to both linear & ...

Describing Function Analysis

→ A signal $y(t)$ is said to be periodic with period T is $y(t+T) = y(t)$ for every value of t.

• The smallest positive value of T for which $y(t+T) = y(t)$, is called fundamental period of $y(t)$.

A periodic signal $y(t)$ may be represented by the series

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad \text{--- (1)}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} y_n \sin(n\omega_0 t + \phi_n) \quad \text{--- (2)}$$

Where $a_n = \frac{2}{T_0} \int_0^{T_0} y(t) \cos n\omega_0 t \cdot dt, \; n = 0, 1, 2 \ldots$ — (3)

$$b_n = \frac{2}{T_0} \int_0^{T_0} y(t) \cdot \sin n\omega_0 t \cdot dt, \quad n = 1, 2 \ldots \quad (4)$$

$$y_n = \sqrt{a_n^2 + b_n^2} \qquad \phi = \tan^{-1}\left(\frac{a_n}{b_n}\right) \quad \text{--- (5)}$$

In eqn (2), the term for $n=1$ is called fundamental or first harmonic and always has the same frequency as the ~~repetition~~ repetation rate of the original periodic waveform.

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Where as $n = 2, 3 \cdots$ gives second, third and fourth harmonic frequencies as integral multiples of the fundamental frequency.

Introducing a change of variable $\psi = \omega_0 t$, obtain the co-efficient of Fourier Series

$$a_n = \frac{1}{\pi} \int_0^{2\pi} y(t) \cos n\omega_0 t \cdot d(\omega_0 t), \quad n = 0, 1, 2 \cdots \quad (6)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} y(t) \sin n\omega_0 t \cdot d(\omega_0 t), \quad n = 1, 2 \cdots \quad (7)$$

Let us assume $x = X \sin \omega t$ with such an i/p, the o/p 'y' of non-linear element will in general be a non-sinusoidal periodic function which may be expressed in terms of Fourier Series

$$y = Y_0 + A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + \cdots$$

The non-linear characteristics, the value Y_0 for all such cases is zero, then

$$y = A_1 \cos \omega t + B_1 \sin \omega t + A_2 \cos 2\omega t + B_2 \sin 2\omega t + \cdots$$

$$y_1 = A_1 \cos \omega t + B_1 \sin \omega t \quad (8)$$

We can write

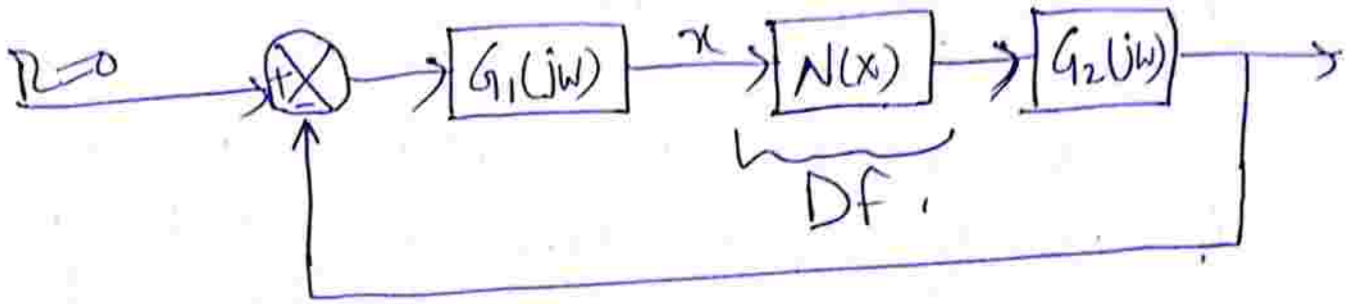
$$y_1(t) = A_1 \sin(\omega t + 90^\circ) + B_1 \sin \omega t$$

$$= Y_1 \sin(\omega t + \phi) \quad (9)$$

The co-efficients A_1 & B_1 of the fourier series

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y\cos\omega t \, d(\omega t)$$

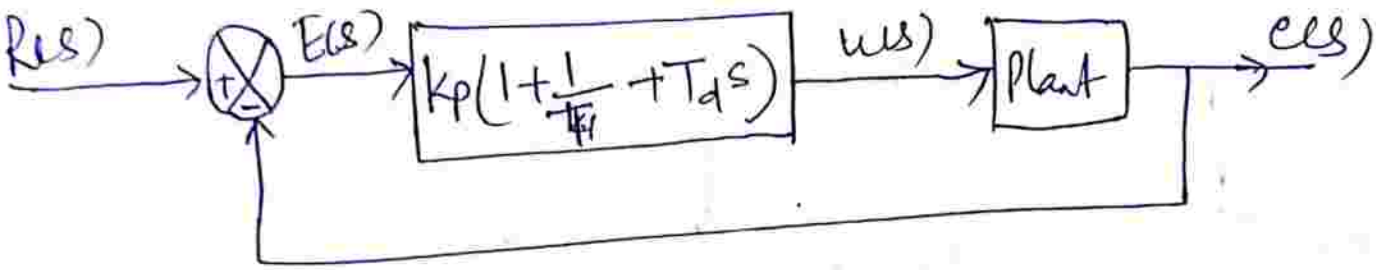
$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y\sin\omega t \, d(\omega t)$$



Tuning of PID Controllers

- PID controllers (compensators) are commonly employed in process control industries.

~~Hence~~ The technique to be adopted for determining the proportional integral and derivative constants of the controller depends upon the dynamic response of the plant.



This error E(s) is manipulated by the controller (PID) to produce a command signal for the plant according to the relationship,

$$U(s) = K_p\left(1+\frac{1}{T_i}s + T_d s\right)$$

$$u(t) = k_p\left[e(t) + \frac{1}{T_i}\int_0^t e(t)\,dt + T_d\frac{de(t)}{dt}\right]$$

in time domain.

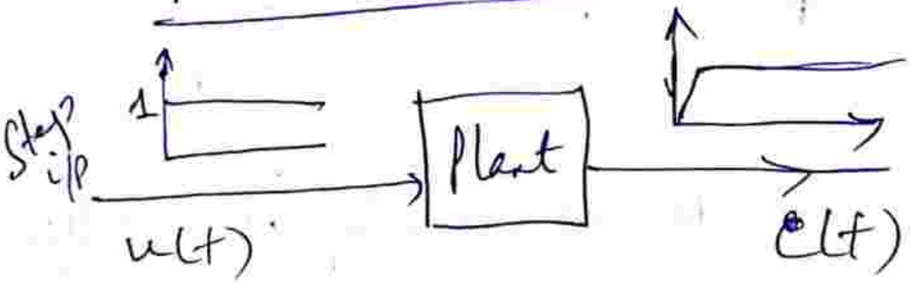
where K_p = Proportional gain
T_i = Integral time constant
T_d = Derivative time constant.

Ziegler-Nicholas Rules

Z-N proposed rules for determining values of proportional gain K_p, integral time T_i and derivative time T_d based on the transient response of a given plant.

* There are two methods
  i) First Method
  ii) Second Method

First Method



Function $\frac{C(s)}{U(s)}$ may then be approximated by a first order syst. with a transportation lag as follows

$$\frac{C(s)}{U(s)} = \frac{K e^{-Ls}}{Ts+1}$$

PID controller tuned by the 1st method of Ziegler-Nicholas rules gives

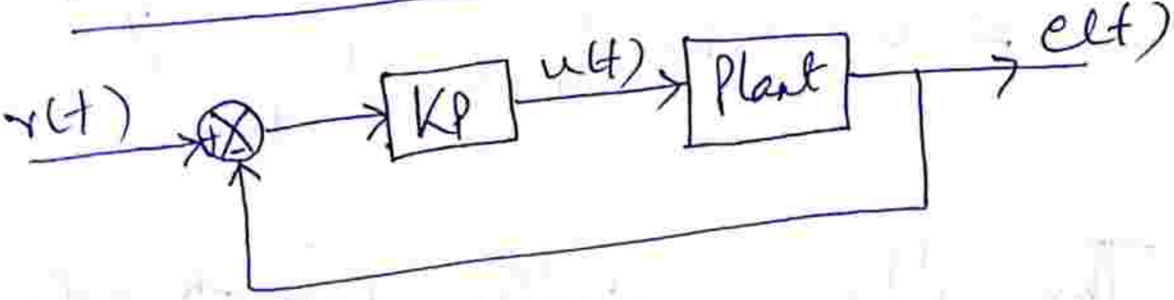
$$G_c(s) = K_p\left(1+\frac{1}{T_i s}+T_d s\right)$$

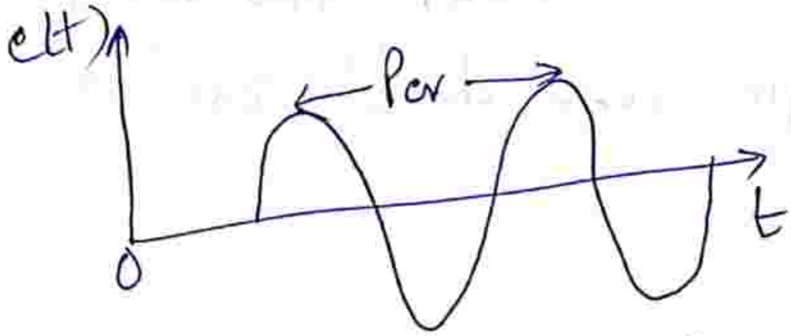
$$= 1.2\frac{T}{L}\left(1+\frac{1}{2L}+0.5L\right)$$

$$G_c(s) = 0.6T \frac{\left(s+\frac{1}{L}\right)^2}{s}$$

* Thus the PID controller has a pole at the origin and double zeros at $s = -1/L$

Second Method





* In the Second Method of Z-N, $T_i = \infty$ & $T_d = 0$. Using proportional control only, increase K_p from 0 to a critical value K_{cr} at which the o/p first exhibits sustained oscillations.

→ So, the critical gain 'K_{cr}' and the corresponding period 'P_{cr}' are experimentally determined.

PID controller tuned by the second method of Z-N rule gives

$$G_c(s) = K_p\left(1+\frac{1}{T_i s}+T_d s\right)$$

$$= 0.6K_{cr}\left(1+\frac{1}{0.5P_{cr}s}+0.125P_{cr}s\right)$$

$$= 0.075K_{cr}P_{cr}\frac{\left(s+\frac{4}{P_{cr}}\right)^2}{s}$$

* The PID controller has pole at the origin and double zeros at $s = -\frac{4}{P_{cr}}$.

Jury's Stability Test

→ This criterion is ~~also~~ algebraic one.
→ The roots of a characteristics polynomial $F(z) = 1+G(z)H(z)$ within the unit circle for determining system stability of a sample data control system.

* The nth order polynomial in 'z'.

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0 \quad a_n > 0$$

i) Necessary conditions for the roots of the above polynomial $F(z)$ to lie within a unit circle in z-plane are

(1) $F(1) > 0$

(2) $(-1)^n F(-1) > 0$

ii) Sufficient conditions for the roots of the above polynomial $F(z)$ to lie within a unit circle in z-plane are determined as follows.

* It consists on $(2n-3)$ ~~arrays~~ rows.

Row	z^0	z^1	z^2	z^3	....	z^n
1	a_0	a_1	a_2	a_3	.....	a_n
2	a_n	a_{n-1}	a_{n-2}	a_{n-3}	- - -	a_0
3	b_0	b_1	b_2			
4						
5						
6						
$2n-3$						

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} \quad \text{for 3rd row}, \; k = 0, 1, \ldots n-1$$

$$C_k = \begin{vmatrix} b_{n-1} & b_k \\ c_0 & c_{n-2-k} \end{vmatrix} \quad \text{for 5th row}, \; k = 0, 1, \ldots n-2$$

$$d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix} \quad \text{for 7th row}, \; k = 0, 1, \ldots (n-3)$$

* Sufficient conditions for the roots of the characteristics polynomial $F(z)$ to lie within a unit circle in z-plane are given by

$|a_0| < a_n \quad |b_0| > |b_{n-1}|$

$|c_0| > |c_{n-2}|$

Ex Determine the stability of a sample-data control system having following characteristic polynomial

$$2z^4 + 8z^3 + 12z^2 + 5z + 1 = 0$$

Sol^n $F(1) = [2(1)^4 + 8(1)^3 + 12(1)^2 + 5(1) + 1] = 28$

$\therefore \quad F(1) > 0$

$(-1)^n F(-1) = 2(-1)^4 + 8(-1)^3 + 12(-1)^2 + 5(-1) + 1$

$= 2 - 8 + 12 - 5 + 1 = 2$

$\therefore (-1)^n F(-1) > 0$

Row	z^0	z^1	z^2	z^3	z^4
1	1	5	12	8	2
2	2	8	12	5	1
3	b_0	b_1	b_2	b_3	
4	b_3	b_2	b_1	b_0	
5	c_0	c_1	c_2		

$2\times4-3$
$=5$

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} \qquad n=4,\quad k=0,1,\dots(n-1)$$

$$k_0;\quad b_0 = \begin{vmatrix} a_0 & a_4 \\ a_4 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1\times1-2\times2=-3$$

$$k=1;\quad b_1 = \begin{vmatrix} a_0 & a_3 \\ a_4 & a_1 \end{vmatrix} = \begin{vmatrix} 1 & 8 \\ 2 & 5 \end{vmatrix} = 1\times5-2\times8 = -11$$

$$k=2;\quad b_2 = \begin{vmatrix} a_0 & a_2 \\ a_4 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & 8 \\ 2 & 8 \end{vmatrix} = 1\times8-2\times8=-8$$

$$k=3\quad b_3 = \begin{vmatrix} a_0 & a_1 \\ a_4 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix} = 1\times8-2\times5=-2$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix} \quad \text{for 5th row} \quad k=0,1,\dots(n-2)$$

$$k_0;\quad c_0 = \begin{vmatrix} b_0 & b_{4-1-0} \\ b_{4-1} & b_0 \end{vmatrix} = \begin{vmatrix} b_0 & b_3 \\ b_3 & b_0 \end{vmatrix} = \begin{vmatrix} -3 & -2 \\ -2 & -3 \end{vmatrix} = 5$$

$$k=1;\quad c_1 = \begin{vmatrix} b_0 & b_{4-1-1} \\ b_{4-1} & b_1 \end{vmatrix} = \begin{vmatrix} b_0 & b_2 \\ b_3 & b_1 \end{vmatrix} = \begin{vmatrix} -3 & -8 \\ -2 & -11 \end{vmatrix} = 17$$

$$k=2;\quad c_2 = \begin{vmatrix} b_0 & b_{4-1-2} \\ b_{4-1} & b_2 \end{vmatrix} = \begin{vmatrix} b_0 & b_1 \\ b_3 & b_2 \end{vmatrix} = \begin{vmatrix} -3 & -11 \\ -2 & -8 \end{vmatrix} = 2$$

Page 4 | Sufficient conditions for stability are

$|a_0| < |a_4|$ or $|1| < |2|$ Satisfied

$|b_0| > |b_3|$ or $|3| > |2|$ - do -

$|c_0| > |c_2|$ or $|5| > |2|$ Satisfied.

* The Required necessary and Sufficient conditions are Satisfied. Hence, the System is Stable.